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A COMBINATORIAL PROOF OF THE BORSUK–ULAM ANTIPODAL POINT THEOREM

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ABSTRACT
We give a proof of Tucker's Combinatorial Lemma that proves the fundamental nonexistence theorem: There exists no continuous map from B^n to S^{n-1} that maps antipodal points of ∂B^n to antipodal points of S^{n-1}.

The combinatorial proof of the Brouwer fixed point theorem due to Sperner is well known. A similar proof was given by A. Tucker [Tu] for the Borsuk–Ulam antipodal point theorem but doesn't appear to be as widely known. Tucker proves his combinatorial lemma in dimension 2 and states that it can be generalized. It appears as an exercise in S. Lefschetz's classic [L, p. 141], and a proof of a generalization was published by Ky Fan [F]. Although reasonably straightforward, I believe that it is worth recording an elementary presentation of this beautiful lemma. The impetus for doing this came from my desire to present the Borsuk–Ulam theorem to an undergraduate class with no background in topology. Several years ago L. Lovász [Lo] solved a combinatorial problem of Kneser by appealing, after a suitable reduction, to the Borsuk–Ulam theorem (see also Baranyi [B] for a simpler proof, also based on the Borsuk–Ulam theorem). It is thus not without interest to point out that a combinatorial proof can be given to this theorem.

TUCKER'S LEMMA. *Let Σ be a triangulation of the cube $[-1, +1]^n = C^n$, centrally symmetric on the boundary of C^n, i.e. for every simplex σ of Σ on ∂C^n, $-\sigma$ is also in Σ. Let φ be a labelling of the vertices of Σ by ele-*

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ments of $\{\pm 1, \pm 2, \ldots, \pm n\}$ *such that for all vertices* $P \in \partial C^n$, $\varphi(P) = -\varphi(-P)$. *Then there are adjacent vertices* P, Q *of* Σ *with*

$$\varphi(P) = -\varphi(Q).$$

Given a continuous map f from C^n to its boundary such that for $P \in \partial C^n$,

$$\varphi(-P) = -\varphi(P),$$

one can triangulate C^n with Σ so that for adjacent vertices $\| f(P) - f(Q) \| < \frac{1}{10}$. Then inducing a labelling $\varphi(P)$ by which a face of ∂C^n $f(P)$ belongs we get a contradiction to Tucker's lemma and thereby prove the basic:

NONEXISTENCE THEOREM. *The exists no continuous map from* B^n *to* S^{n-1} *that maps antipodal points on* ∂B^n *to antipodal points of* S^{n-1}.

It is straightforward to deduce "fixed point" theorems from this basic nonexistence theorem, and since these procedures are well known I won't go into further detail. Suffice it to say that these arguments are easy exercises in analysis and don't require any deeper knowledge of the topological structure of the sphere and the projective space obtained by identifying antipodal points. The remainder of the paper is devoted to an exposition of a proof of Tucker's lemma.

Denote by C^+_{n-1} the positive half of ∂C^n; explicitly C^+_{n-1} consists of those points $(x_1, \ldots, x_n)$ such that $x_i = +1$ for some i. Next let C^+_{n-2} denote the "positive half" of $\partial(C^+_{n-1})$. Note that ∂C^+_{n-1} consists of $n(n-1)$, $(n-2)$-cubes defined by the requirements $x_i = +1$, $x_j = -1$ for some pair $i \neq j$. For the positive half of ∂C^+_{n-1} we have to select from each pair $\{x_i = +1, x_j = -1\}$, $\{x_i = -1, x_j = +1\}$ a single cube, and we always take that pair where $i < j$, $x_i = +1$, $x_j = -1$. It is easy to check that in this way we get an $(n-2)$-cell. In general C^+_{n-j} will denote the union of $(n-j)$-cells defined by j constraints of the type

$$x_{a_1} = +1, \quad x_{a_2} = -1, \quad x_{a_3} = +1, \quad \ldots, \quad x_{a_j} = (-1)^{j+1}$$

for some $i \leqq a_1 < a_2 < \cdots < a_j \leqq n$. It is easy to check that C^+_{n-j} is an $(n-j)$ manifold whose boundary is $C^+_{n-j-1} \cup (-C^+_{n-j-1})$. Assuming now that we are given a triangulation of C^n and a labelling of the vertices with labels drawn from $\{\pm 1, \pm 2, \ldots, \pm n\}$ satisfying the conditions of the lemma but that *no edge labelled* $\{+i, -i\}$ *exists*, we will proceed to derive a contradiction.

Our basic calculation will be to count the number of pairs in C^+_k

{($k+1$)-simplex; face labelled $(i_1, -i_2, i_3, \dots, (-1)^{k-1} i_k\}$ for $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n$, modulo 2 in two different ways, once by summing over ($k+1$)-simplices and once by summing over k-simplices. Then summing the two expressions over all k-tuples $(i_1, \dots, i_k)$ will give us an equality between a certain expression on C_k^+ and the same expression on C_{k-1}^+. At the extremes, C_1^+ and C_{n-1}^+, we shall get different answers and that will be our contradiction. We shall start at C_1^+ and do two steps in detail, then explain the general case and C_n^+. It will be convenient to have also C_0^+, the positive boundary of C_1^+, which consists of *one* point.

Let, in general, $S_k(i_0, i_1, \dots, i_k)$ denote the number of k-simplices in C_k^+ that are labelled $i_0, i_1, \dots, i_k$. In C_1^+ count pairs {1-simplex, vertex labelled i} by edges to obtain

$$2S_1(i,i) + \sum_{j \neq i} S_1(i,j) + S_1(i,-j).$$

Note that by our hypotheses $S_1(i,-i) = 0$. Then count the same set by summing over vertices to obtain

$$2 \times \{\text{interior vertices of } C_1^+, \text{ labelled } i\} + S_0(i) + S_0(-i).$$

The last two terms come from the fact that $\partial C_1^+ = C_0^+ \cup (-C_0^+)$ and the hypotheses on the antisymmetry of the labelling on ∂C^n. Modulo 2, after summing over all i we obtain the basic equality

$$\sum_{i<j} S_1(i,-j) + S_1(-i,j) = \sum_i S_0(i) + S_0(-i). \tag{1}$$

Note that the terms with $S_i(+i,+j)$ dropped out modulo 2 since they appear once in the expression for i and once in the expression for j. The right-hand side of (1) equals 1. Count modulo 2 the number of pairs in C_2^+ {2-simplex, 1-face labelled $(i,-j)$}, $i<j$ by summing over 2-simplices in C_2^+ to get

$$\sum_{k \neq i,j} S_2(i,-j,k) + S_2(i,-j,-k).$$

Count the same collection modulo 2 by summing over 1-simplices labelled $(i,-j)$ to get

$$S_1(i,-j) + S_1(-i,j).$$

Once again the interior 1-simplices drop out because we count modulo 2. Thus we have

$$\sum_{k \neq i,j} S_2(i, -j, k) + S_2(i, -j, -k) = S_1(i, -j) + S_1(-i, j).$$

Summing over all pairs $i < j$, and reducing again modulo 2, we obtain

$$\sum_{i_0 < i_1 < i_2} S_2(i_0, -i_1, i_2) + S_2(-i_0, i_1, -i_2) = \sum_{i<j} S_1(i, -j) + S_1(-i, j). \tag{2}$$

Note that the right-hand side of (2) is exactly the left-hand side of (1). In general we shall obtain

$$\begin{aligned} &\sum_{i_1 < i_1 < \cdots < i_k} S_k(i_0, -i_1, \ldots, (-1)^k i_k) + S_k(-i_0, i_1, -i_2, \ldots, (-1)^{k+1} i_k) \\ (k) \qquad &= \sum_{i_0 < i_1 < \cdots < i_{k-1}} S_{k-1}(i_0, -i_1, \ldots, (-1)^{k-1} i_{k-1}) \\ &\qquad + S_{k-1}(-i_0, i_1, \ldots, (-1)^k i_{k-1}). \end{aligned}$$

Before explaining the general case let's see what happens at $(n-1)$. There is only one n-tuple of distinct labels, $(1, -2, \ldots, \pm n)$. At this point, count in C^n (the full n-cube) the number of incidences in the triangulation of an n-simplex with a face labelled $(1, -2, 3, \ldots, \pm n)$. Since there are no further labels, and no edge $(i, -i)$ exists, the only types of labellings of n-simplices that occur have *two equal labels* and thus, modulo 2, this sum is *zero*, when calculated over n-simplices. On the other hand, calculating by faces labelled $(1, -2, \ldots, \pm n)$ gives exactly

$$S_{n-1}(1, -2, \ldots, \pm n) + S_{n-1}(-1, +2, \ldots, \mp n)$$

which by equations (1)–$(n-1)$ equals 1 modulo 2. This is the desired contradiction.

The proof of (k) is exactly like the special cases treated. The key observation is that any expression of the type

$$S_k(i_0, \ldots, i_a, i_{a+1}, \ldots, i_k)$$

that we encounter where the signs of i_a, i_{a+1} are the same will occur twice in the grand sum that is the left-hand side of (k).

In conclusion we should remark that this method of proof can give quantitative versions of the Borsuk–Ulam theorem. Quite precise such results appear in [DS] but they are based on the Borsuk theorems and don't give an "elementary" proof of the basic theorems.

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